Lecture 15: Tree

Definition 1 Let G be a graph. A vertex v of G is called a cut-vertex if $G - v$ has more components than G.

Theorem 1 *Let G be a connected graph with $|G| \geq 2$ and let $v \in V(G)$.*

a: If $\deg(v) = 1$, then $G - v$ is connected, so that v is never a cut-vertex.

b: If $G - v$ is connected, then either $\deg(v) = 1$ or v is on a cycle.

Proof: (a) Let $a, b \in V(G - v)$, $a \neq b$. Since G is connected there is a $a - b$ path in G. Eveidently the vertex v can not be the internal vertex of this path, as the degree of internal vertex is ≥ 2. So the path $a - b$ is available in $G - v$. So, $G - v$ is connected.

(b) Assume $G - v$ is connected. If $\deg(v) = 1$, then nothing to prove. So let $\deg(v) \geq 2$. To show that v is on a cycle in G. Let u and w be two distinct neighbors of v. Since $G - v$ is connected, there is a path $(u = u_1, u_2, \ldots, u_k = w)$ in $G - v$. Then $(u = u_1, u_2, \ldots, u_k = w, v)$ is a cycle.

Definition 2 Let G be a graph. An edge e in G is called a cut-edge or a bridge if $G - e$ has more connected components than that of G.

Proposition 1 *Let G be connected and let $e = uv$ be a cut-edge. Then $G - e$ has two components, one containing u and the other containing v.*

Proof: If $G - e$ is not disconnected, then by definition, e can not be a cut edge. So $G - e$ has at least two components. Let G_u (respectively, G_v) be the component containing the vertex u (respectively, v). We claim that these are the only components.

Let $w \in V(G)$. Since G is connected, there is a path, say P, from w to u. Moreover, either P contains v as its internal vertex or P does not contain v. In the first case, $w \in V(G_v)$ and in the latter case, $w \in V(G_u)$. Thus, every vertex of G is either in $V(G_v)$ or in $V(G_u)$ and hence the required result follows.

Theorem 2 *Let G be a graph and let e be an edge. Then, e is a cut-edge iff e is not on a cycle.*

Proof: Suppose $e = uv$ is a cut-edge of G. Let F be the component of G that contains e. Then, by the above Proposition, $F - e$ has two components, namely, F_u that contains u and F_v that contains v.

Let if possible, $C = (u, v = v_1, \ldots, v_k = u)$ be a cycle containing $e = uv$. Then $(v = v_1, \ldots, v_k = u)$ is a $u - v$ path in $F - e$. Hence, $F - e$ is still connected, a contradiction. Thus, e cannot be on any cycle.

Conversely, let $e = uv$ be an edge which is not on any cycle. Now, suppose that F is the component of G that contains e. We need to show that $F - e$ is disconnected. Let if possible, there is a $u - v$ path, say $(u = u_1, \ldots, u_k = v)$, in $F - e$. Then, $(v, u = u_1, \ldots, u_k = v)$ is a cycle containing e. A contradiction to e not lying on any cycle.

Definition 3 A connected graph G with no cycles is called a tree. A collection of trees is called a forest.

Proposition 2 *A tree on n vertices has $n - 1$ edges.*

Proof: We apply induction on n. Take a tree on $n \geq 2$ vertices and delete an edge e. Then, we get two subtrees T_1, T_2 of order n_1, n_2, respectively, where $n_1 + n_2 = n$. So, $E(T) = E(T_1) \cup E(T_2) \cup \{e\}$. By induction hypothesis $|E(T)| = |E(T_1)| + |E(T_2)| + 1 = n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 = n - 1$.

Corollary 1 *A tree with at least two vertices has at least two pendant vertices.*

Proof: Let T be a tree on $n \geq 2$ vertices. Then $\sum_{v \in V(T)} \deg(v) = 2|E(T)| = 2n - 2$. Then it is easy to see that T has at least two vertices of degree 1 (by PHP).

We now prove that the following statements that characterize trees are equivalent.

Theorem 3 *Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Then f.s.a.e.*

1. *G is a tree.*
2. *Let $u, v \in V$. Then there is a unique path from u to v.*
3. *G is connected and $n = m + 1$.*

Proof: $(1 \Rightarrow 2)$: Since G is connected, for each $u, v \in V$, there is a path from u to v. On the contrary, let us assume that there are two distinct paths P_1 and P_2 that join the vertices u and v. Since P_1 and P_2 are distinct and both start at u and end at v, there exist vertices, say u_0 and v_0, such that the paths P_1 and P_2 take different edges after the vertex u_0 and the two paths meet again at the vertex v_0 (note that u_0 can be u and v_0 can be v). In this case, we see that the graph G has a cycle consisting of the portion of the path P_1 from u_0 to v_0 and the portion of the path P_2 from v_0 to u_0. This contradicts the assumption that G is a tree (it has no cycle).

$(2 \Rightarrow 3)$: Since for each $u, v \in V$, there is a path from u to v, the connectedness of G follows. We need to prove that $n = m + 1$. We prove this by induction on the number of vertices of a graph. The result is clearly true for $n = 1$ or $n = 2$. Let the result be true for all graphs that have n or less than n vertices. Now, consider a graph G on $n + 1$ vertices that satisfies the conditions of Item 2. The uniqueness of the path implies that if we remove an edge, say $e \in E$, then the graph G will become disconnected. That is, $G \setminus e$ will have exactly two components. Let the number of vertices in the two components be n_1 and n_2. Then $n_1, n_2 \leq n$ and $n_1 + n_2 = n + 1$. Hence, by induction hypothesis, the number of edges in $G - e$ equals $(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2 = n - 1$ and hence the number of edges in G equals $n - 1 + 1 = n$. Thus, by the principle of mathematical induction, the result holds for all graphs that have a unique path from each pair of vertices.

$(3 \Rightarrow 1)$: It is already given that G is a connected graph. We need to show that G has no cycle. So, on the contrary, let us assume that G has a cycle of length k. Then this cycle has k vertices and k edges. Now, consider the $n - k$ vertices that do not lie of the cycle. Then for each vertex (corresponding to the $n-k$ vertices), there will be a distinct edge incident with it on the smallest path from the vertex to the cycle. Hence, the number of edges will be greater than or equal to $k + (n - k) = n$. A contradiction to the assumption that the number of edges equals $n - 1$. Thus, the required result follows.